DEDEKIND'S TRANSPOSITION PRINCIPLE FOR LATTICES OF EQUIVALENCE RELATIONS

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Abstract. We prove a version of Dedekind's Transposition Principle that holds in lattices of equivalence relations.

In this note we prove a version of Dedekind's Transposition Principle[1] that holds in all (not necessarily modular) lattices of equivalence relations. Let X be a set and let $\operatorname{Eq} X$ denote the lattice of equivalence relations on X. Given $\alpha, \beta \in \operatorname{Eq} X$, we define the *interval sublattice of equivalence relations above* α *and below* β, denoted $[\![\alpha, \beta]\!]$, as follows:

$$[\![\alpha, \beta]\!] := \{\gamma \in \operatorname{Eq} X \mid \alpha \leqslant \gamma \leqslant \beta\}.$$

Let L be a sublattice of $\operatorname{Eq} X$. Given $\alpha, \beta \in L$, let $[\![\alpha, \beta]\!]_L := [\![\alpha, \beta]\!] \cap L$, which we call an *interval sublattice of* L, or more simply, an *interval of* L. Given $\alpha, \beta, \theta \in L$, let $[\![\alpha, \beta]\!]_L^\theta$ denote the set of equivalence relations in the interval $[\![\alpha, \beta]\!]_L$ that permute with θ. That is,

$$[\![\alpha, \beta]\!]_L^\theta := \{\gamma \in L \mid \alpha \leqslant \gamma \leqslant \beta \text{ and } \gamma \circ \theta = \theta \circ \gamma\}.$$

Lemma 1. *If* $\eta, \theta \in L \leqslant \operatorname{Eq} X$, *and if* $\eta \circ \theta = \theta \circ \eta$, *then*

$$[\![\theta, \eta \vee \theta]\!]_L \cong [\![\eta \wedge \theta, \eta]\!]_L^\theta \leqslant [\![\eta \wedge \theta, \eta]\!]_L.$$

The lemma states that the sublattice $[\![\theta, \eta \vee \theta]\!]_L$ is isomorphic to the lattice, $[\![\eta \wedge \theta, \eta]\!]_L^\theta$, of relations in L that are below η, above $\eta \wedge \theta$, and permute with θ; moreover, $[\![\eta \wedge \theta, \eta]\!]_L^\theta$ is a sublattice of $[\![\eta \wedge \theta, \eta]\!]_L$. To prove this, we need the following generalized version of *Dedekind's Rule*:[2]

Lemma 2. *If* $\alpha, \beta, \gamma \in L \leqslant \operatorname{Eq} X$, *and if* $\alpha \leqslant \beta$, *then we have the following identities of subsets of* X^2*:*

$$\alpha \circ (\beta \cap \gamma) = \beta \cap (\alpha \circ \gamma), \tag{1}$$

$$(\beta \cap \gamma) \circ \alpha = \beta \cap (\gamma \circ \alpha). \tag{2}$$

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[1]If L is a modular lattice, then for any two elements $a, b \in L$ the intervals $[\![b, a \vee b]\!]$ and $[\![a \wedge b, a]\!]$ are isomorphic. See [1], or [2, page 57].

[2]In the group theory setting, the well known Dedekind's Rule states that if A, B, C are subgroups of a group, and $A \leqslant B$, then we have the following identity of sets: $A(B \cap C) = B \cap AC$.

Proof. We prove (1); the proof of (2) is similar. First we check that $\alpha\circ(\beta\cap\gamma)\subseteq\beta\cap(\alpha\circ\gamma)$. Indeed, since $\alpha\leqslant\beta$, we have

$$\alpha\circ(\beta\cap\gamma)\subseteq\alpha\vee(\beta\cap\gamma)\leqslant\beta\vee(\beta\cap\gamma)=\beta.$$

Also, $\beta\cap\gamma\leqslant\gamma$ implies $\alpha\circ(\beta\cap\gamma)\subseteq\alpha\circ\gamma$. Therefore, $\alpha\circ(\beta\cap\gamma)\subseteq\beta\cap(\alpha\circ\gamma)$.

For the reverse inclusion, fix $(x,y)\in\beta\cap(\alpha\circ\gamma)$. Since $(x,y)\in\alpha\circ\gamma$, there exists $c\in X$ such that $x\,\alpha\,c\,\gamma\,y$. We must produce $d\in X$ such that $x\,\alpha\,d\,(\beta\cap\gamma)\,y$. In fact, $d=c$ works, since $(x,c)\in\alpha\leqslant\beta$ implies $c\,\beta\,x\,\beta\,y$, so $(c,y)\in\beta\cap\gamma$. □

Proof of Lemma 1. Let $\eta,\theta\in L\leqslant\operatorname{Eq}X$ be permuting equivalence relations in L, so $\eta\circ\theta=\theta\circ\eta=\eta\vee\theta$. Consider the mapping $\varphi:[\![\theta,\eta\vee\theta]\!]_L\to[\![\eta\wedge\theta,\eta]\!]$ given by $\alpha\mapsto\alpha\wedge\eta$. Clearly φ maps $[\![\theta,\eta\vee\theta]\!]_L$ into the sublattice $[\![\eta\wedge\theta,\eta]\!]_L\leqslant[\![\eta\wedge\theta,\eta]\!]$. Moreover, it's easy to see that the range of φ consists of elements of L that permute with θ, so that φ maps into $[\![\eta\wedge\theta,\eta]\!]_L^{\theta}$. Indeed, if $\alpha\in[\![\theta,\eta\vee\theta]\!]_L$, then by Lemma 2 we have $(\alpha\wedge\eta)\circ\theta=\alpha\cap(\eta\circ\theta)=\alpha\cap(\theta\circ\eta)=\theta\circ(\alpha\wedge\eta)$.

Next, consider the mapping $\psi:[\![\eta\wedge\theta,\eta]\!]_L^{\theta}\to[\![\theta,\eta\vee\theta]\!]$ given by $\psi(\alpha)=\alpha\circ\theta$. Note that $\psi(\alpha)=\alpha\circ\theta=\alpha\vee\theta$, an element of L, since the domain of ψ is a set of relations in L that permute with θ. We show that the two maps

$$\varphi:[\![\theta,\eta\vee\theta]\!]_L\ni\alpha\longmapsto\alpha\wedge\eta\in[\![\eta\wedge\theta,\eta]\!]_L^{\theta}\tag{3}$$

$$\psi:[\![\eta\wedge\theta,\eta]\!]_L^{\theta}\ni\alpha\longmapsto\alpha\circ\theta\in[\![\theta,\eta\vee\theta]\!]_L.\tag{4}$$

are inverse lattice isomorphisms. It is clear that these maps are order preserving. Also, for $\alpha\in[\![\theta,\eta\vee\theta]\!]_L$ we have, by Lemma 2, $\psi\,\varphi(\alpha)=(\alpha\wedge\eta)\circ\theta=\alpha\cap(\eta\circ\theta)=\alpha\cap(\eta\vee\theta)=\alpha$. For $\alpha\in[\![\eta\wedge\theta,\eta]\!]_L^{\theta}$, we have, by Lemma 2, $\varphi\,\psi(\alpha)=\varphi(\alpha\circ\theta)=(\alpha\circ\theta)\wedge\eta=\alpha\circ(\theta\wedge\eta)$.

To complete the proof of Lemma 1, we show that $[\![\eta\wedge\theta,\eta]\!]_L^{\theta}$ is a sublattice of $[\![\eta\wedge\theta,\eta]\!]_L$. Fix $\alpha,\beta\in[\![\eta\wedge\theta,\eta]\!]_L^{\theta}$. We show

$$(\alpha\vee\beta)\circ\theta\subseteq\theta\circ(\alpha\vee\beta),\tag{5}$$

and

$$(\alpha\wedge\beta)\circ\theta\subseteq\theta\circ(\alpha\wedge\beta).\tag{6}$$

The reverse inclusions follow by symmetric arguments.

Fix $(x,y)\in(\alpha\vee\beta)\circ\theta$. Then there exist $c\in X$ and $n<\omega$ such that $x\,(\alpha\circ^{(n)}\beta)\,c\,\theta\,y$. Thus, $(x,y)\in\alpha\circ^{(n)}\beta\circ\theta$. Since θ permutes with both α and β, we have $(x,y)\in\theta\circ\alpha\circ^{(n)}\beta\subseteq\theta\circ(\alpha\vee\beta)$, which proves (5). Fix $(x,y)\in(\alpha\wedge\beta)\circ\theta$. Then $(x,y)\in(\alpha\circ\theta)\cap(\beta\circ\theta)=(\theta\circ\alpha)\cap(\theta\circ\beta)$. Therefore, there exist d_1, d_2 such that $x\,\theta\,d_1\,\alpha\,y$ and $x\,\theta\,d_2\,\beta\,y$. Note that $(d_1,y)\in\alpha\leqslant\eta$ and $(d_2,y)\in\beta\leqslant\eta$, so $(d_1,d_2)\in\eta$. Also, $d_1\,\theta\,x\,\theta\,d_2$, so $(d_1,d_2)\in\theta$. Therefore, $(d_1,d_2)\in\eta\wedge\theta\leqslant\alpha\wedge\beta$. In particular, $d_1\,\beta\,d_2\,\beta\,y$, so $(d_1,y)\in\alpha\wedge\beta$. Thus, $x\,\theta\,d_1\,(\alpha\wedge\beta)\,y$, which proves (6).

□

References

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